A Rigorous and Efficient Method of Moments Solution for Curved Waveguide Bends

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***Abstract*—An accurate and computationally efficient method of moments solution together with a mode-matching technique for the analysis of curved bends in a general parallel-plate waveguide is presented. In order to exemplify the techniques, the method is applied to study the transmission characteristics of single and cascaded curved E- and H-plane bends in a rectangular waveguide. It is shown that the effect of the orientation of cascaded bends on the transmission properties can be significant, and examples to demonstrate this effect are included. Results of the convergence with increasing number of expansion functions illustrate that only a few terms need to be considered for accurate evaluation of the transmission characteristics of structures having single and multiple bends. Comparison with measurements for single and cascaded curved H-plane bends in a WR-90 waveguide show good agreement with the predicted result.**

I. Introduction

Numerous efforts have been made over the last few decades to solve for the propagation characteristics including modal solutions of curved waveguide bends e.g., [1]–[8]. In 1948 Rice [1] formulated a matrix solution for curved rectangular waveguides by expanding the transverse eigensolutions in sine and cosine functions. He then used a limiting process to obtain approximate solutions for bends with large radius of curvature. Lewin [2] showed that the analysis of rectangular waveguides containing curved E- and H-plane bends can be reduced to that of a corresponding parallel-plate waveguide if the dielectric medium inside the waveguide is homogeneous. Lewin then derived approximate modal solutions for curved E- and H-plane bends in a rectangular waveguide by means of a perturbation analysis. Utilizing approximate formulas for Bessel functions, Cochran and Pecina [3] solved the appropriate characteristic equations for the propagating modes in a curved waveguide. In a recent publication, Accadino and Bertin [4] extended the approach of Cochran and Pecina by transforming the ill-conditioned characteristic equations for the evanescent modes in a curved rectangular waveguide into stable ones, and calculated the reflection coefficient of curved E-plane bends.

Curved waveguide bends have also been analyzed in related areas such as acoustics and the recently emerged field of nanostructure physics e.g., [5]. For example, Furnell and Bies [6] formulated a Ritz–Rayleigh variational procedure to analyze a curved bend in an acoustical duct. Sols and Macucci [7] as well as Lent [8], utilizing a finite-element method, have analyzed curved bends in quantum waveguide structures with ballistic quasi one-dimensional electron transport.

The methods described above are either approximate or are computationally intensive. In comparison, a rigorous and computationally efficient method of moments analysis of curved waveguide bends is presented in this paper. Here, the derivations are given for curved bends in a general parallel-plate waveguide which can be applied to various types of waveguide configurations such as rectangular waveguides, quantum waveguides, and microstrips (using the waveguide model [9]).

The theoretical part of this paper begins with the derivation of the modal solutions for the curved bend by means of the method of moments [10]. These waveguide modes are utilized to construct modal expansions in the curved bend region. Mode-matching is then employed to obtain the scattering parameters of a single bend discontinuity. The scattering parameters of cascaded bends are computed from the single bend result by utilizing the generalized scattering matrix technique [11], [12]. The method described here is validated by comparing the computed results with measured data obtained for single and cascaded curved H-plane bends in a rectangular waveguide, and its computational efficiency is demonstrated by means of convergence studies.

II. Modal Solutions

Fig. 1 shows a general parallel-plate waveguide containing a curved bend discontinuity. The plates are either perfect electric or magnetic conductors (electric or magnetic walls). The dielectric medium inside the waveguide is assumed to be constant (homogeneous case). This waveguide structure independently supports TE- and TM-modes (with respect to the longitudinal direction, i.e., z- or s-direction) [2]. The solutions of the TE- and TM-modes in the parallel-plate waveguides (regions I and II) are readily obtained from the scalar wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi(x, z) = 0; \qquad k^2 = \omega^2\mu\epsilon \quad (1)$$

which is satisfied by the corresponding transverse electric

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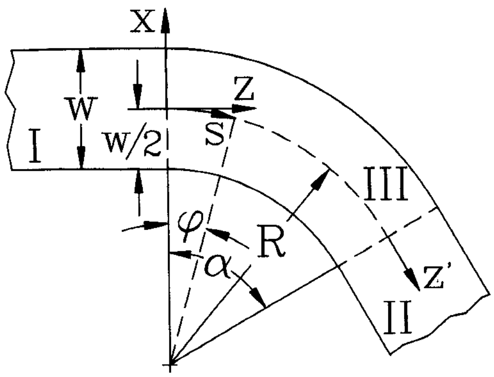

Fig. 1. Geometry of a curved bend in a parallel-plate waveguide.

component, E_y, or transverse magnetic field component, H_y, respectively [13], [14]. These modal solutions have the general form

$$\psi_n(x, z) = \phi_n(x) e^{\pm j\beta_n z}. \tag{2}$$

The function $\phi_n(x)$ is the normalized transverse eigensolution and $\beta_n = [k^2 - (n\pi/w)^2]^{1/2}$ is the phase constant of mode n. Depending on the type of mode (TE or TM) and the kind of wall (electric or magnetic) considered, the boundary conditions at the plates are either of Dirichlet or Neumann type. Hence, $\phi_n(x)$ are either normalized sine or cosine functions which form a complete orthonormal set of eigenfunctions [13], [14].

The modal solutions in the curved region are conveniently found from the wave equation given in the curved coordinate system (x, y, $s = R\varphi$) [2], [9] where R is the center radius of the bend (see Fig. 1).

$$\left(u^2 \frac{\partial^2}{\partial x^2} + u \frac{\partial}{\partial x} + R^2 \frac{\partial^2}{\partial s^2} + k^2u^2\right) \psi(x, s) = 0,$$

$$u = x + R. \tag{3}$$

Inserting the ansatz

$$\psi(x, s) = f(x) e^{\pm j\tilde{\beta} s} \tag{4}$$

into (3) results in the eigenvalue equation

$$Lf = \left[u \frac{d}{dx}\left(u \frac{d}{dx}\right) + k^2u^2\right] f = R^2\tilde{\beta}^2 f \tag{5}$$

with real, discrete eigenvalues $\lambda_n = R^2\tilde{\beta}_n^2$ and real eigenfunctions $f_n(x)$ corresponding to mode n [15].

In order to convert the eigenvalue problem given in (5) into an equivalent matrix eigenvalue equation by means of the method of moments [10], the inner product

$$\langle g, h\rangle = \int_{-w/2}^{w/2} \frac{1}{u} g(x) h(x)\, dx \tag{6}$$

with weighting function $1/u$ is defined. This choice of inner product makes the differential operator L together with the given boundary conditions self-adjoint [10], [15]. According to the method of moments, the transverse field solution $f(x)$ is expanded into a set of basis functions $b_i(x)$ as

$$f(x) = \sum_i d_i b_i(x) \tag{7}$$

and the inner product of the eigenvalue equation with an appropriately selected set of testing functions $t_j(x)$ is taken. Here, Galerkin's procedure is employed where the transverse eigensolutions $\phi_n(x)$ of the corresponding parallel-plate waveguide are conveniently chosen as basis and testing functions ($b_i = t_i = \phi_i$). This choice of basis and testing functions considerably simplifies the mode-matching procedure as seen below. The resulting matrix eigenvalue equation with eigenvalues $\tilde{\beta}_n^2$ and eigenvectors $\boldsymbol{d}_n = (d_1^n\ d_2^n\ d_3^n \cdots)^T$, where T represents the transpose, is found to be (Appendix I):

$$(k^2\boldsymbol{P} - \boldsymbol{S})\boldsymbol{d}_n = \tilde{\beta}_n^2 \boldsymbol{Q}\boldsymbol{d}_n \tag{8a}$$

with

$$P_{ij} = \int_{-w/2}^{w/2} \frac{u}{R} \phi_i \phi_j\, dx \tag{8b}$$

$$S_{ij} = \int_{-w/2}^{w/2} \frac{u}{R} \frac{d\phi_i}{dx} \frac{d\phi_j}{dx}\, dx \tag{8c}$$

$$Q_{ij} = \int_{-w/2}^{w/2} \frac{R}{u} \phi_i \phi_j\, dx. \tag{8d}$$

It should be noted that the matrices $\boldsymbol{P}$, $\boldsymbol{S}$, and $\boldsymbol{Q}$ are all real and symmetric. Since in (8d) the term $1/u > 0$, it is easily shown that $\boldsymbol{Q}$ is also positive definite [16]. Hence, all eigenvalues $\tilde{\beta}_n^2$ are real and distinct, and the corresponding eigenvectors $\boldsymbol{d}_n$ are orthonormal with respect to matrix $\boldsymbol{Q}$ [15], that is

$$\boldsymbol{D}^T\boldsymbol{Q}\boldsymbol{D} = \boldsymbol{I}, \qquad \boldsymbol{D} = (\boldsymbol{d}_1\ \boldsymbol{d}_2 \cdots \boldsymbol{d}_n \cdots) \tag{9}$$

It should be added here that the procedure given above can be generalized and applied to curved bends in a parallel-plate waveguide filled with an inhomogeneous dielectric medium [17]. However, for the inhomogeneous case the parallel-plate waveguide results cannot be directly translated to the corresponding rectangular waveguide structure due to coupling of the TE- and TM-modes and possible existence of complex waves [16], [18].

III. Mode-Matching Solution

The general solution of the transverse field components in the different regions of the curved waveguide bend discontinuity can be expressed in terms of infinite series of the waveguide modes derived in the previous section. The modal amplitudes in the different regions may be found from a normal mode-matching procedure applied to the continuity of the transverse field components at the interfaces between the curved and the two straight regions. The mode-matching procedure can be simplified by applying a generalization of Kühn's superposition procedure [19] to the curved region [9], [20]. More specifically, the superposition solution of the transverse field in the curved region is constructed by alternately extending the boundary condition at the plates (electric or magnetic walls) along the interfaces at $s = 0$ and $s = R\alpha$ [9], [20]. The actual implementation of this procedure is somewhat different for TE- and TM-modes and also depends on the type of boundary condition prevailing at the plates (electric or magnetic walls). In the following, the mode-matching procedure is described for TE-modes in a curved bend configuration bounded by electric walls. The corresponding results for the magnetic wall case as well as for TM-modes are given in Appendix II. Likewise, only the homogeneous case is treated in this paper. The generalization to the inhomogeneous parallel-plate waveguide case can be implemented by using a similar procedure as the one given in [21] for a step discontinuity in a graded-index dielectric slab waveguide.

The normalized modal expansions of the transverse field components for TE-modes in region I are given as

$$E_y^I = \sum_k (a_k^I e^{-j\beta_k z} + b_k^I e^{j\beta_k z}) \sqrt{Z_k}\, \phi_k \tag{10a}$$

$$H_x^I = -\sum_k (a_k^I e^{-j\beta_k z} - b_k^I e^{j\beta_k z}) \sqrt{Y_k}\, \phi_k \tag{10b}$$

where $Z_k = Y_k^{-1} = \omega\mu_0/\beta_k$ is the characteristic wave impedance of mode k. Similarly, the modal expansions in region II can be expressed as

$$E_y^{II} = \sum_k (a_k^{II} e^{j\beta_k z'} + b_k^{II} e^{-j\beta_k z'}) \sqrt{Z_k}\, \phi_k \tag{11a}$$

$$H_x^{II} = \sum_k (a_k^{II} e^{j\beta_k z'} - b_k^{II} e^{-j\beta_k z'}) \sqrt{Y_k}\, \phi_k. \tag{11b}$$

As mentioned above, a superposition solution is used in the curved region (region III) where an electric wall is first placed at the interface $s = 0$ and then at $s = R\alpha$.

$$E_y^{III} = \sum_k c_k^a \sin(\tilde{\beta}_k s) f_k + \sum_k c_k^b \sin[\tilde{\beta}_k (s - R\alpha)] f_k. \tag{12}$$

The corresponding transverse magnetic field component H_x is obtained from the relationship $j\omega\mu_0 H_x = R/u \partial E_y/\partial s$ as

$$H_x^{III} = \frac{1}{j\omega\mu_0} \sum_k c_k^a \tilde{\beta}_k \cos(\tilde{\beta}_k s) \frac{R}{u} f_k + \frac{1}{j\omega\mu_0} \sum_k c_k^b \tilde{\beta}_k \cdot \cos[\tilde{\beta}_k (s - R\alpha)] \frac{R}{u} f_k. \tag{13}$$

The continuity condition for the transverse field components across the interfaces between the curved region and the two straight regions leads to four independent equations. The two equations relating E_y are solved for the modal coefficients c_n^a and c_n^b by multiplying each equation by $R/u f_n(x)$ and integrating over the waveguide width. Here, the orthogonality property of the transverse eigensolutions $f_n(x)$ with respect to the weighting function R/u as given by (9) is utilized. Expressing the modal amplitudes in terms of column vectors, the resulting set of equations is given in matrix notation as

$$\boldsymbol{c}^a = \boldsymbol{T}_s^{-1} \boldsymbol{D}^T \boldsymbol{Q} \boldsymbol{Z}^{1/2} (\boldsymbol{a}^{II} + \boldsymbol{b}^{II}) \tag{14a}$$

$$\boldsymbol{c}^b = -\boldsymbol{T}_s^{-1} \boldsymbol{D}^T \boldsymbol{Q} \boldsymbol{Z}^{1/2} (\boldsymbol{a}^{I} + \boldsymbol{b}^{I}) \tag{14b}$$

where $\boldsymbol{Z}^{1/2}$ is a diagonal matrix with $Z_{nn}^{1/2} = (Z_n)^{1/2}$, and $\boldsymbol{T}_s$ is a diagonal matrix with diagonal elements $\sin(\tilde{\beta}_n R\alpha)$. Matrices $\boldsymbol{D}$ and $\boldsymbol{Q}$ have been defined in (9) and (8d), respectively. The two remaining equations relating H_x along the interfaces are solved for the modal amplitudes in the straight regions by multiplying each equation by $\phi_n(x)$ followed by an integration over the waveguide width where the orthogonality property of the eigenfunctions $\phi_n(x)$ is utilized. The resulting set of equations is expressed in matrix notation as

$$\boldsymbol{a}^I - \boldsymbol{b}^I = j\boldsymbol{Z}^{1/2} \boldsymbol{Q} \boldsymbol{D} \tilde{\boldsymbol{Y}} (\boldsymbol{c}^a + \boldsymbol{T}_c \boldsymbol{c}^b) \tag{15a}$$

$$\boldsymbol{a}^{II} - \boldsymbol{b}^{II} = -j\boldsymbol{Z}^{1/2} \boldsymbol{Q} \boldsymbol{D} \tilde{\boldsymbol{Y}} (\boldsymbol{T}_c \boldsymbol{c}^a + \boldsymbol{c}^b). \tag{15b}$$

Here, $\tilde{\boldsymbol{Y}}$ is a diagonal matrix with $\tilde{Y}_{nn} = \tilde{\beta}_n/\omega\mu_0$, and $\boldsymbol{T}_c$ is a diagonal matrix with diagonal elements $\cos(\tilde{\beta}_n R_\alpha)$. Notice that since the eigenfunctions of the parallel-plate waveguide are used as basis and testing functions, no additional integration needs to be performed in the mode-matching procedure. In the next step, the modal coefficients $\boldsymbol{c}^a$ and $\boldsymbol{c}^b$ are eliminated from (14) and (15). This procedure results in two matrix equations relating the modal amplitudes in the two straight regions. Finally, the scattering parameters for the curved bend discontinuity are readily derived from these two matrix equations [9]. From the scattering parameters of a single bend discontinuity, the scattering parameters of waveguide structures containing multiple bends are then obtained by applying the generalized scattering matrix (GSM) technique [11], [12]. In contrast to earlier applications of the GSM technique, the orientation of two cascaded bends is here carefully taken into consideration.

IV. Results

As part of the verification procedure for the method presented in this paper, the reflection coefficient of an S-shaped E-plane bend (Fig. 2(a)) and corresponding single E-plane bend (U-shaped bend) (Fig. 2(b)) in a rectangular waveguide has been evaluated and compared with the computed results of Accatino and Bertin [4] as shown in Fig. 2(c). The results for S- and U-shaped bends shown in Fig. 2(c) indicate that it may be critical how two bends are cascaded. In Fig. 3, two cascaded curved H-plane bends are studied where a straight waveguide section of length L is inserted in between. The orientation of the cascaded bends becomes less significant with increasing length L and is negligible for lengths greater than the waveguide width over which the evanescent modes excited at the junctions are sufficiently attenuated.

The computational efficiency of the method presented in this paper is demonstrated in Fig. 4 where the reflection coefficient of the S-shaped bend is plotted as function of

(a)

(b)

(c)

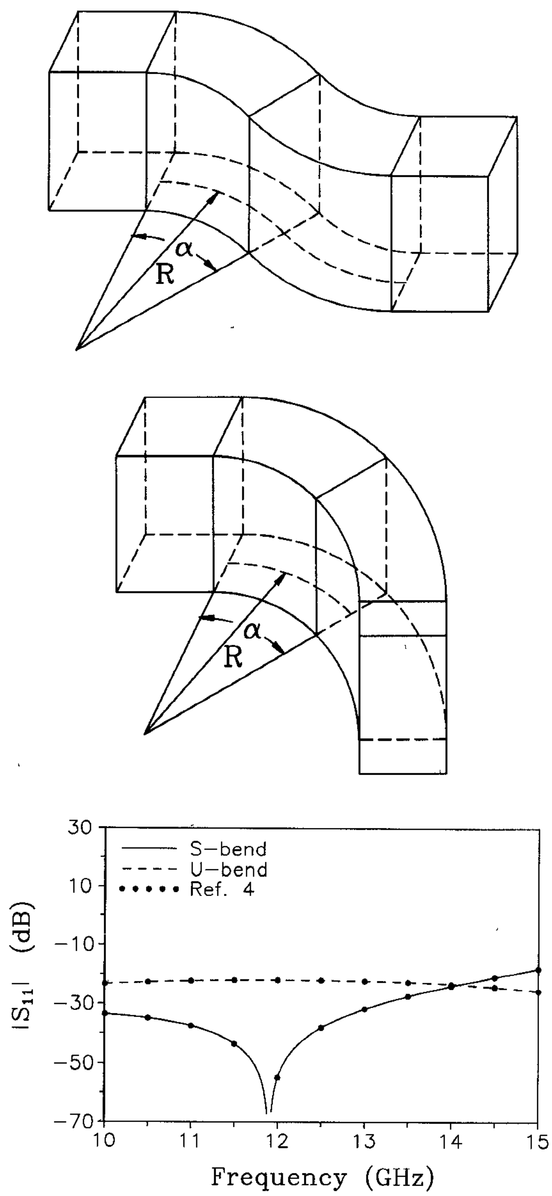

Fig. 2. Cascaded 45° E-plane bends with R = 8 mm in a WR-75 rectangular waveguide: (a) S-bend; (b) U-bend; (c) reflection coefficient.

the number of basis functions retained in the method of moments procedure. For purpose of simplicity, the number of terms used in the expansions for the transverse eigensolutions and in the modal expansions are kept the same in all calculations shown in this paper. It can be seen from Fig. 4 that only a few expansion terms are needed for accurate solutions. A typical analysis for the S-bend structures shown in Fig. 3(a)–(b) with 5 expansion terms and 100 data points takes approximately 4.5 minutes on a 286 based personal computer (12 MHz) and 8 seconds on an HP Apollo 425T workstation.

In order to validate the computational method presented here, S-parameter measurements on U- and S-shaped H-plane bends in a WR-90 waveguide have been performed. The waveguide structures together with two straight waveguides and a short plate which were used as a TRL calibration set, have been built in-house. The mea-

(a)

(b)

(c)

(d)

(e)

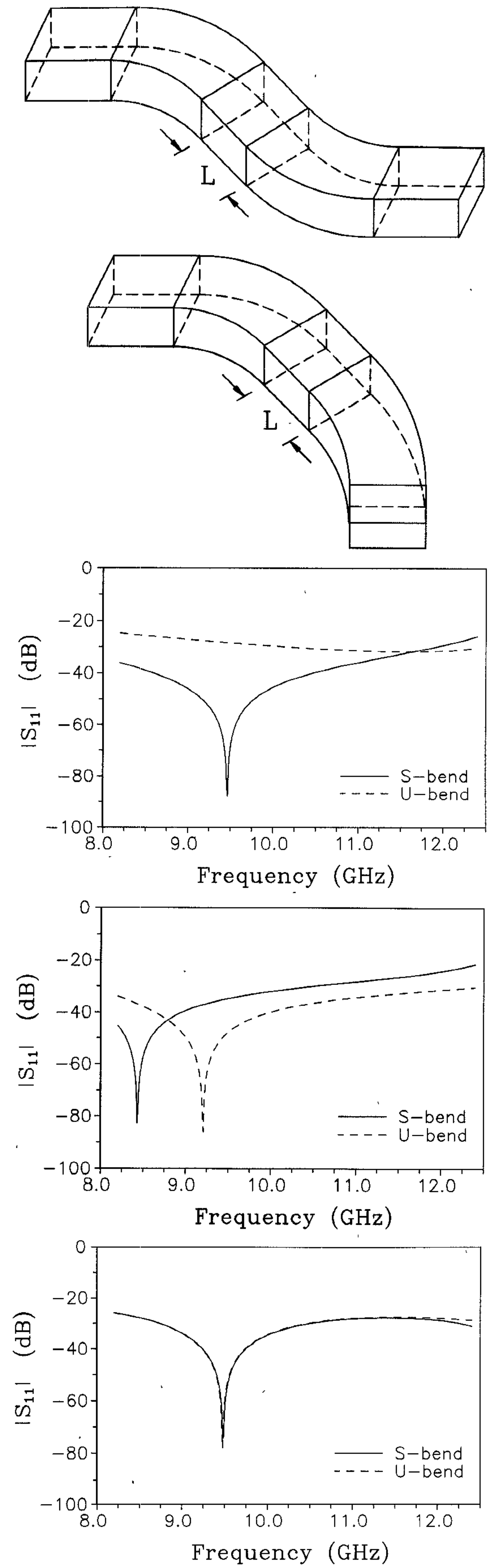

Fig. 3. Cascaded 30° H-plane bends with R = 15.24 mm in a WR-90 rectangular waveguide: (a) S-type bend; (b) U-type bend; reflection coefficient for (c) L = 0, (d) L = 5 mm, and (e) L = 25 mm.

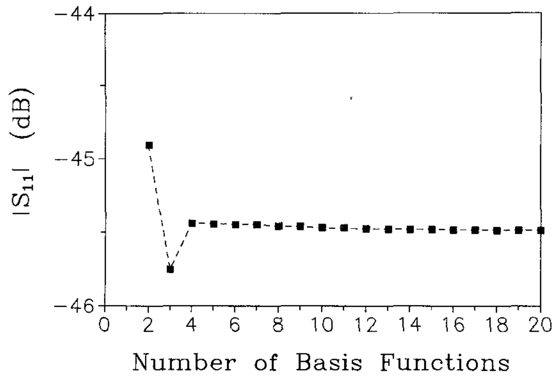

Fig. 4. Reflection coefficient of an S-shaped H-plane bend configuration (Fig. 3(c)) as function of the number of basis function at $f = 10$ GHz.

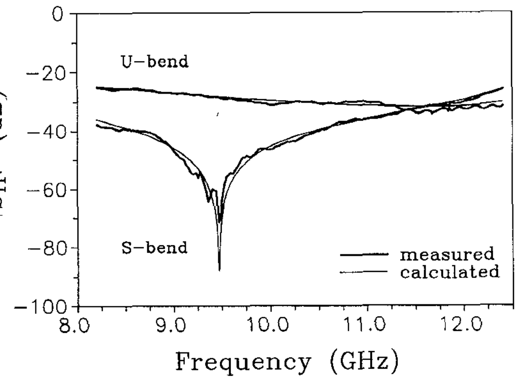

Fig. 5. Comparison of the measured data with the calculated reflection coefficient of the S-shaped H-plane bend given in Fig. 3(c).

surements were performed on an HP 8510B network analyzer. The resonance apparent in the measured data for the S-shaped bend is correctly predicted by the simulation, and an overall good agreement between the measured data and simulated results is found as seen in Fig. 5.

V. Conclusions

An accurate and efficient method for analyzing single and multiple curved bend waveguide discontinuities has been described and applied to rectangular waveguides. Computed results for transmission properties of single and cascaded curved E- and H-plane bends have been presented. Measurements on cascaded curved H-plane bends in a WR-90 waveguide were in good agreement with the predicted results. The computed results for the scattering parameters converge very rapidly with increasing number of expansion functions and modes used in the method of moments and mode-matching analysis, respectively. Only a few expansion terms and modes need to be considered for accurate solutions so that the method described here is computationally efficient and can be implemented on a 286 based personal computer. The method described in this paper is quite general and has also been applied to curved bends in other types of waveguide structures such as microstrip bends based on the waveguide model and single and multiple bends in quantum waveguide structures as shown in [22].

Appendix I
Conversion of the Operator Eigenvalue Equation into an Equivalent Matrix Eigenvalue Equation

In this Appendix, matrix eigenvalue equation (8) is derived. First, the expansion for $f(x)$ given by (7) is inserted into operator eigenvalue equation (5):

$$L \sum_i d_i \phi_i(x) = \sum_i d_i u \frac{d}{dx}\left(u \frac{d}{dx}\phi_i(x)\right) + k^2 \sum_i d_i u^2 \phi_i(x) = \tilde{\beta}^2 R^2 \sum_i d_i \phi_i(x) \tag{A1}$$

Multiplying (A1) by the testing function $\phi_j(x)$ and integrating over the waveguide width with respect to weighting function $1/u$ gives

$$\sum_i d_i \int_{-w/2}^{w/2} \frac{d}{dx}\left(u \frac{d}{dx}\phi_i(x)\right)\phi_j(x)\,dx + k^2 \sum_i d_i \int_{-w/2}^{w/2} u\phi_i(x)\phi_j(x)\,dx = \tilde{\beta}^2 R^2 \sum_i d_i \int_{-w/2}^{w/2} \frac{1}{u}\phi_i(x)\phi_j(x)\,dx. \tag{A2}$$

The first integral in (A2) can be simplified by an integration by parts and application of the boundary conditions (Dirichlet or Neumann). Using the definitions given in (8b)–(8d), the final result can be written as

$$\sum_i d_i(-S_{ij} + k^2 P_{ij}) = \tilde{\beta}^2 \sum_i d_i Q_{ij} \tag{A3}$$

which is equivalent to the matrix equation (8a) where the expansion coefficients d_i have been combined into the column vector $\boldsymbol{d}$.

Appendix II
Mode-Matching Solutions

A. TE Modes—Magnetic Wall Case

The superposition solution for TE modes in the case of magnetic walls is constructed by alternately extending the magnetic walls across the two interfaces. The resulting transverse electric field in the curved region reads

$$E_y^{III} = \sum_i c_k^a \cos(\tilde{\beta}_k s) f_k + \sum_k c_k^b \cos[\tilde{\beta}_k (s - R\alpha)] f_k. \tag{A4}$$

The continuity equations for the transverse magnetic field component H_x are solved for the modal coefficients c_n^a and c_n^b by multiplying each equation by $f_n(x)$ and integrating over the waveguide width. The result is given in matrix notation as

$$\boldsymbol{c}^a = -j\tilde{\boldsymbol{Z}}\boldsymbol{T}_s^{-1}\boldsymbol{D}^T\boldsymbol{Y}^{1/2}(\boldsymbol{a}^{II} - \boldsymbol{b}^{II}) \tag{A5}$$

$$\boldsymbol{c}^b = -j\tilde{\boldsymbol{Z}}\boldsymbol{T}_s^{-1}\boldsymbol{D}^T\boldsymbol{Y}^{1/2}(\boldsymbol{a}^{I} - \boldsymbol{b}^{I}) \tag{A6}$$

where the various matrices are defined in Sections II and III. The continuity equations for E_y are solved for the modal amplitudes in the straight regions by multiplication by $\phi_n(x)$ and integration over the waveguide width. The resulting set of equations is expressed in matrix notation as

$$a^I + b^I = Y^{1/2}D(c^a + T_c c^b) \tag{A7}$$

$$a^{II} + b^{II} = Y^{1/2}D(T_c c^a + c_b). \tag{A8}$$

B. TM Modes—Electric Wall Case

The superposition solution for TM modes in the case of electric walls is constructed by alternately extending the electric walls across the two interfaces. The resulting transverse magnetic field in the curved region reads

$$H_y^{III} = \sum_k c_k^a \cos(\tilde{\beta}_k s) f_k + \sum_k c_k^b \cos[\tilde{\beta}_k(s - R\alpha)] f_k. \tag{A9}$$

The corresponding transverse electric field component E_x is readily obtained from the relationship

$$E_x^{III} = \frac{j\omega\mu_0}{k^2}\frac{R}{u}\frac{\partial H_y^{III}}{\partial x}. \tag{A10}$$

The modal coefficients c_n^a and c_n^b are obtained by multiplying the continuity equations for E_x by $f_n(x)$ and integrating over the waveguide width. The resulting matrix equations read

$$c^a = j\tilde{Y}T_s^{-1}D^T Z^{1/2}(a^{II} + b^{II}) \tag{A11}$$

$$c^b = -j\tilde{Y}T_s^{-1}D^T Z^{1/2}(a^I + b^I) \tag{A12}$$

where $Z^{1/2}$ is a diagonal matrix with $Z_{nn}^{1/2} = (Z_n)^{1/2}$, $Z_n = Y_n^{-1} = \omega\mu_0\beta_n/k^2$, and $\tilde{Y}$ is a diagonal matrix with $\tilde{Y}_{nn} = (\omega\mu_0\tilde{\beta}_n/k^2)^{-1}$. The continuity equations for H_y are multiplied by $\phi_n(x)$ and integrated to

$$a^I - b^I = Z^{1/2}D(c^a + T_c c^b) \tag{A13}$$

$$a^{II} - b^{II} = -Z^{1/2}D(T_c c^a + c_b). \tag{A14}$$

C. TM Modes—Magnetic Wall Case

In order to construct the superposition solution for TM modes in the case of magnetic walls, the boundary condition is alternatively extended across the two interfaces. The resulting transverse magnetic field in the curved region reads

$$H_y^{III} = \sum_k c_k^a \sin(\tilde{\beta}_k s) f_k + \sum_k c_k^b \sin[\tilde{\beta}_k(s - R\alpha)] f_k. \tag{A15}$$

The electric field can be derived using expression (A10). The continuity equations for the transverse magnetic field component H_y are solved for the modal coefficients c_n^a and c_n^b by multiplying each equation by $R/uf_n(x)$ and integrating over the waveguide width. The result is given in matrix notation as

$$c^a = -T_s^{-1}D^T Q Y^{1/2}(a^{II} - b^{II}) \tag{A16}$$

$$c^b = -T_s^{-1}D^T Q Y^{1/2}(a^I - b^I) \tag{A17}$$

where $Y^{1/2}$ is a diagonal matrix with $Y_{nn}^{1/2} = (Y_n)^{1/2}$, $Y_n = Z_n^{-1} = (\omega\mu_0\beta_n/k^2)^{-1}$. The continuity equations for E_x are solved for the modal amplitudes in the straight regions by multiplication by $\phi_n(x)$ and integration over the waveguide width. The resulting set of equations is expressed in matrix notation as

$$a^I + b^I = jY^{1/2}QD\tilde{Z}(c^a + T_c c^b) \tag{A18}$$

$$a^{II} + b^{II} = jY^{1/2}QD\tilde{Z}(T_c c^a + c^b) \tag{A19}$$

where the diagonal matrix $\tilde{Z}$ is defined as $\tilde{Z}_{nn} = \omega\mu_0\tilde{\beta}_n/k^2$.

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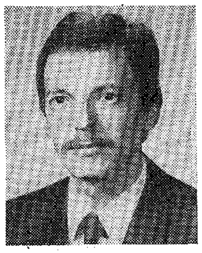
References

[1] S. O. Rice, "Reflections from circular bends in rectangular waveguides-matrix theory," *Bell Syst. Tech. J.*, vol. 27, no. 2, pp. 305–349, 1948.
[2] L. Lewin, *Theory of Waveguides*. New York: Wiley, 1975.
[3] J. A. Cochran and R. G. Pecina, "Mode propagation in continuously curved waveguides," *Radio Science*, vol. 1, no. 6, pp. 679–696, 1966.
[4] L. Accatino and G. Bertin, in *Proc. 20th Europ. Microwave Conf.*, Budapest, Sept. 1990, pp. 1246–1250.
[5] *Proc. IEEE, Special Issue on Nanoelectronics*, Aug. 1991.
[6] G. D. Furnell and D. A. Bies, "Characteristics of modal wave propagation within longitudinally curved acoustic waveguides," *J. Sound Vibration*, vol. 130, no. 3, pp. 405–423, 1989.
[7] F. Sols and M. Macucci, "Circular bends in electron waveguides," *Phys. Rev. B*, vol. 41, no. 17, pp. 11887–11891, 1990.
[8] C. S. Lent, "Transmission through a bend in an electron waveguide," *Appl. Phys. Lett.*, vol. 56, no. 25, pp. 2554–2556, 1990.
[9] A. Weisshaar and V. K. Tripathi, "Perturbation analysis and modeling of curved microstrip bends," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 1449–1454, 1990.
[10] R. F. Harrington, *Field Computation by Moment Methods*. New York: MacMillan, 1968.
[11] T. Itoh, "Generalized scattering matrix technique," in *Numerical Techniques for Microwave and Millimeter-Wave Passive Structures*, T. Itoh, Ed., New York: Wiley, 1989.
[12] A. Weisshaar, J. Lary, S. M. Goodnick, and V. K. Tripathi, "Analysis and modeling of quantum waveguide structures and devices," *J. Appl. Phys.*, vol. 70, pp. 355–366, July 1991.
[13] N. Marcuvitz, *Waveguide Handbook*. Lexington, MA: Boston Technical Publishers, 1964.
[14] R. E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960.
[15] C. R. Wylie, *Advanced Engineering Mathematics*. 4th ed., New York: McGraw-Hill, 1975.
[16] M. Mrozowski and J. Mazur, "Matrix theory approach to complex

waves," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 781–785, Apr. 1992.
[17] A. Weisshaar, "Method of moments analysis of curved inhomogeneous planar waveguide structures," to be published.
[18] J. Strube and F. Arndt, "Rigorous hybrid-mode analysis of the transition from rectangular waveguide to shielded dielectric image guide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 391–401, May 1985.
[19] E. Kühn, "A mode-matching method for solving field problems in waveguide and resonator circuits," *Arch. Elek. Übertragung.*, vol. 27, pp. 511–513, Dec. 1973.
[20] A. Weisshaar, "Generalized modal analysis of electromagnetic- and quantum-waveguide structures and discontinuities," Ph.D. dissertation, Oregon State University, Mar. 1991.
[21] A. Weisshaar and V. K. Tripathi, "Modal analysis of step discontinuities in graded-index dielectric slab waveguides," *J. Lightwave Technol.*, vol. 10, pp. 593–602, May 1992.
[22] A. Weisshaar, S. M. Goodnick, and V. K. Tripathi, "A rigorous method of moments solution for curved waveguide bends and its applications," *IEEE MTT-S Int. Microwave Symp. Dig.*, pp. 975–978, Albuquerque, NM, June 1992.



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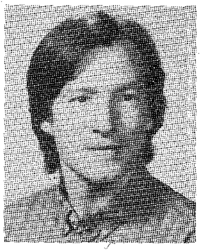

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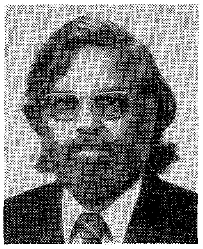

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